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  Of course, we first need to know:

- Which elements in a C*-algebra are linear combinations of projections?

What's known in $B(H)$

- Fillmore (1967) Every operator in $B(H)$ is a linear combination of 257 projections. Pearcy & Topping (1967), Paszkiewicz (1980), Matsumoto (1984) reduced the number to 10 projections.

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- Fong & Murphy (1985): This is the only exception.

What's known in W*-algebras

- Pearcy and Topping (1967), Fack&De La Harpe (1980), Goldstein&Paszkiewicz (1992): all elements in a W*-algebra are linear combination of projections iff the algebra has no finite type I direct summand with infinite dimensional center.

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- Bikchentaev (2005) Every positive invertible element in a W*-algebra without finite type I direct summands with infinite dimensional center is a positive combination of projections.

More recent in W*-algebras

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- Type II_∞ factors (or finite direct sums): if either R_b is finite or b is not in the Breuer ideal of compact operators. *Similar to* $B(H)$.
- "Large center": the central essential spectrum must be bounded away from 0.

What's known in C*-algebras

The following **unital simple** C*-algebras are the span of their projections (mostly work by Marcoux (1998-2010)):

- purely infinite C*-algebras;
- with proper projections but no tracial states;
- real rank zero with unique tracial state satisfying strict comparison of projections ($\tau(p) < \tau(q) \Rightarrow p \prec q$);
- AF-algebras, AT-algebras, or AH-algebras (if with bounded dimension growth) of real rank zero and finitely many extremal tracial states.

The purely infinite case-PCP

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- *Every positive element of the multiplier $\mathcal{M}(\mathcal{A})$ is PCP.*

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Theorem (KNZ, P-AMS (2012))

If $K_0(\mathcal{A})$ is a torsion group and $b \in \mathcal{A}^+$, $\|b\| > 1$ then b is a finite sum of projections.

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- the tracial state space $T(\mathcal{A})$ is non-empty and has finitely many extreme points; (recall that $T(\mathcal{A})$ is convex and w*-cpt);
- strict comparison of projections:

$$\tau(p) < \tau(q) \quad \forall \tau \in T(\mathcal{A}) \;\Rightarrow\; p \precsim q.$$

Finite C*-algebras: linear combinations

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$\mathcal{A}$ a C*-algebra with the listed properties/

Theorem

$\mathcal{A}$ is the linear span of it projections with "control on the coefficients". That is, there is a constant V_0 s.t. for every $b \in \mathcal{A}$, $\exists \lambda_j \in \mathbb{C}, p_j \in \mathcal{A}$ projections s.t

$$b = \sum_1^n \lambda_j p_j \quad \textit{and} \quad \sum_1^n |\lambda_j| \leq V_0 \|b\|.$$

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Question

If $\mathcal{A}$ is the span of its projections, does control of the coefficients follow automatically?

Why control of the coefficients?

Lemma (proof as in Fong & Murphy's (1985) for $B(H)$)

If a C-algebra $\mathcal{A}^+$ is the span of it projections with control on the coefficients and has $RR(\mathcal{A}) = 0$, then every positive invertible is PCP.*

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Beyond invertibles:

Lemma

Let $\mathcal{A}$ have the property that positive invertibles in any corner $r\mathcal{A}r$ are PCP. If $b := \alpha p \oplus a$ with $\alpha > \|a\|$ and $a = qaq \geq 0$, $q \precsim p$, then b is PCP.

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This lemma is the essential tool for attacking the general PCP problem.

First step: commutators

Theorem

If $b \in \mathcal{A}$ and $\tau(b) = 0 \quad \forall \tau \in T(\mathcal{A})$, then b is the sum of 2 commutators (with control on their norms.)

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Theorem
If $b \in \mathcal{A}$ *and* $\tau(b) = 0 \quad \forall \tau \in T(\mathcal{A})$*, then* b *is the sum of 2 commutators (with control on their norms.)*

This theorem holds even when $card(Ext(T(\mathcal{A})) = \infty$.

Ingredients in the proof

- Embed in $\mathcal{A}$ a unital simple AH-algebra $\mathcal{C}$ with real rank zero and dimension growth bounded by 3 and same K-invariants (Lin (2001), Elliott& Gong, Gong (1996, 1997,1998)).
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  (Here is the only place where we use separability.)
- Extend the Fack (1982), Thomsen (1994) construction to this inductive limit case so to approximate b by a bounded number of commutators.
- Use the Marcoux (2002, 2006) machinery to express b as the sum of commutators and then reduce their number to two.
  (Still keep control on the norms.)

From commutators to projections

- Marcoux (2002) proved that if in a C*-algebra there exist three mutually orthogonal projections p_1, p_2 and p_3 such that $1 = p_1 + p_2 + p_3$ and $p_i \precsim 1 - p_i$ for $1 \leq i \leq 3$, then every commutator is a linear combination of 84 projections, with control on the coefficients. (Commutators = sums of certain nilpotents of order two=sums of idempotents = (by Davidson) =linear combinations of projections)

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  This condition is easily satisfied in our case. Thus so far we have:

- every $b \in \mathcal{A}$ s.t. $\tau(b) = 0$ for every tracial state τ is a linear combination of projections with control on the coefficients.

Beyond zero trace

- If there is a unique tracial state τ, then $b = \tau(b)1 + (b - \tau(b)1)$ is a linear combination of projections (just one...) plus a zero-trace element.

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- Using the density of $K_o(\mathcal{A})$ in the continuous affine functions on $T(\mathcal{A})$ (Blackadar (1982)) we get:

Lemma

If card$(Ext(T(\mathcal{A})) < \infty$ then every element in $\mathcal{A}$ is the sum of linear combination of projections plus an element in the kernel of all the traces.

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- These 3 steps conclude the proof. To recap:
  $b=$ linear combination of projections $+ c$, $\tau(c) = 0 \forall \tau in T(\mathcal{A})$;
  $c = [x_1, y_1] + [x_2, y_2]$;
  $[x_i, y_i]=$ linear combination of projections;

and all that with control of the coefficients.

Infinitely many extremal traces?

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The proof mimics the one that a Hamel basis of an infinite separable Banach space cannot be countable.

Finite nonunital C*-algebras: obstruction to PCP

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- The condition is also sufficient. But first, we need the PCP result.

Finite C*-algebras: N&S condition for PCP

Theorem

Let $\mathcal{A}$ be σ-unital, with all properties as above and $card(Ext(T(\mathcal{A})) < \infty$. Then $b \in \mathcal{A}^+$ is PCP if and only if $\bar{\tau}(R_b) < \infty \ \forall \tau \in T(\mathcal{A})$.(Always true if $\mathcal{A}$ is unital.)

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Corollary

With $\mathcal{A}$ as above, $b \in \mathcal{A}$ is a linear combination of projections in $\mathcal{A}$ if and only if $\bar{\tau}(R_b) < \infty \ \forall \tau \in T(\mathcal{A})$.

Ingredients in the proof, part I

We can work in a corner where the "identity is not too far from the range projection".

Lemma

If $\bar{\tau}(R_b) < \infty \; \forall \tau \in T(\mathcal{A})$ *then there is a trace preserving isomorphism*

$$\Psi : her(b) \to \Psi(her(b)) \subset r\mathcal{A}r \text{ for some } r \in \mathcal{A}, \; \tau(r) < 2\bar{\tau}(R_b).$$

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Why solving PCP question first? Notice that

- decomposing $\Psi(b)$ into a PCP in $r\mathcal{A}r$, *necessarily* in $\Psi(\mathrm{her}(b))$ gives a PCP decomposition of b;
- decomposing $\Psi(b)$ into a linear combination of projections in $r\mathcal{A}r$ does not yield a decomposition of b.

Ingredients in the proof, part II

- Previous lemma permits to embed b into a unital algebra so that $\bar{\tau}(N_b) < \bar{\tau}(R_b) \; \forall \tau \in T(\mathcal{A})$.

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- Use the key lemma that we have seen before:

Lemma

Let $\mathcal{A}$ have the property that positive invertibles in any corner $r\mathcal{A}r$ are PCP. If $b := \alpha p \oplus a$ with $\alpha > \|a\|$ and $a = qaq \geq 0$, $q \precsim p$, then b is PCP.

- Plus more work - the proof is technical.

THANK YOU FOR YOUR ATTENTION